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A Note on Integral Transforms and Differential Equations

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ABSTRACT

In this work a new integral transform, namely Sumudu transform was applied to solve linear ordinary differential equations with constant coefficients with convolution terms. Further in order to generate a pde with non constant coefficients, the convolutions were used and solutions were demonstrated.

Keywords: Sumudu transform, differential equations and convolution.

1. INTRODUCTION

In the literature there are several works on the theory and applications of integral transforms such as Laplace, Fourier, Mellin, Hankel, to name a few, but very little on the power series transformation such as Sumudu transform, probably because it is little known and not widely used yet. The Sumudu transform was proposed originally by Watugala (1993) to solve differential equations and control engineering problems.

In Watugula (2002), the Sumudu transform was applied for functions of two variables. Some of the properties were established by Weerakoon (1994, 1998). In Asiru (2002), further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in Kadem (2005). In fact it was shown that there is strong relationship between Sumudu and other integral transform, see Kilicman *et al.* (2011). In particular the relation between Sumudu transform and Laplace transforms was proved in Kilicman (2011).

Further, in Eltayeb *et al.* (2010), the Sumudu transform was extended to the distributions and some of their properties were also studied in Kilicman and Eltayeb (2010). Recently, this transform is applied to solve the system of differential equations, see Kilicman *et al.* (2010).

Note that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except the factor *n*, see Zhang (2007). Thus if $f(t)=\sum_{n=0}^{\infty} a_n t^n$ then $F(u)=\sum_{n=0}^{\infty} n! a_n u^n$, see Kilicman et al. (2011). Similarly, the Sumudu transform sends combinations, C(m, n), into permutations, P(m,n) and hence it will be useful in the discrete systems.

The Sumudu transform is defined by the formula

$$F(u)=S\left[f(t);\ u\right]$$

$$=\frac{1}{u}\int_0^{\infty} e^{-\left(\frac{t}{u}\right)} f(t)\ dt, \quad u\in\left(-\tau_1,\ \tau_2\right).$$

over the set of

$$A=\left\{ f(t) \left| \begin{array}{l} \exists M, \text{ and or/, } \tau_2>0, \text{ such that } \left|f(t)\right|<Me^{\frac{|t|}{\tau_j}} \\ \text{if } t\in(-1)_j\times[0,\infty) \end{array} \right. \right\}.$$

Our purpose in this study is to show the applicability of this interesting new transform and its efficiency in solving the linear ordinary differential equations with constant and non constant coefficients having the non homogenous term as convolutions.

Throughout this paper we need the following theorem which was given by Belgacem (2007), where they discussed the Sumudu transform of the derivatives:

Theorem 1: Let $n \geq 1$ and let $G_n(u)$ be the Sumudu transforms of the $f^{(n)}(t)$. Then

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}$$

for more details, see Belgacem (2007).

Since the transform is defined as improper integral therefore we need to discuss the existence and the uniqueness.

Theorem 2: Let $f(t)$ and $g(t)$ be continuous functions defined for $t \geq 0$ and have Sumudu transforms, $F(u)$ and $G(u)$, respectively. If $F(u) = G(u)$ then $f(t) = g(t)$ where u is complex number.

Proof: If α are sufficiently large, then the integral representation of f by

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\frac{t}{u}} F(u)\, du$$

since $F(u) = G(u)$ almost everywhere then we have

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\frac{t}{u}} G(u)\, du.$$

By using Laplace transform of the function $f(t)$ denoted by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

can be rewritten after a change of variable, $w = st$ with $dw = sdt$

$$F(s) = \int_0^\infty e^{-w} f\left(\frac{w}{s}\right) \frac{dw}{s},$$

from above relation and replace u by $\frac{1}{s}$ we obtain the inverse Sumudu transform as follow

$$f(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{st}G\left(\frac{1}{s}\right)\frac{ds}{s}=g(t),$$

and the theorem is proven.

In the next theorem we study the existence of Sumudu transform as follows.

Theorem 3: (Existence of the Sumudu transform) If f is of exponential order, then its Sumudu transform $S\left[f(t);u\right]=F(u)$ is given by

$$F(u)=\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}f(t)dt,$$

where $\frac{1}{u}=\frac{1}{\eta}+\frac{i}{\tau}$. The defining integral for F exists at points $\frac{1}{u}=\frac{1}{\eta}+\frac{i}{\tau}$ in the right half plane $\frac{1}{\eta}>\frac{1}{k}$ and $\frac{1}{\zeta}>\frac{1}{L}$.

Proof: Using $\frac{1}{u}=\frac{1}{\eta}+\frac{i}{\tau}$ and we can express $F(u)$ as

$$F(u)=\int_0^\infty f(t)\ \cos\ \left(\frac{t}{\tau}\right)e^{-\frac{t}{\eta}}dt$$

$$-i\int_0^\infty f(t)\ \sin\ \left(\frac{t}{\tau}\right)e^{-\frac{t}{\eta}}dt.$$

Then for values of $\frac{1}{\eta}>\frac{1}{k}$, we have

$$\int_0^\infty |f(t)|\left|\cos\left(\frac{t}{\tau}\right)\right|e^{-\frac{t}{\eta}}dt\le M\int_0^\infty e^{\left(\frac{1}{K}-\frac{1}{\eta}\right)t}dt$$

$$\le\left(\frac{M\eta K}{\eta-K}\right)$$

and

$$\int_0^\infty |f(t)|\left|\sin\left(\frac{t}{\tau}\right)\right| e^{-\frac{t}{\eta}} dt \leq M\int_0^\infty e^{\left(\frac{1}{K}-\frac{1}{\eta}\right)t} dt$$
$$\leq \left(\frac{M\eta K}{\eta - K}\right)$$

which imply that the integrals defining the real and imaginary parts of F exist for value of $\mathrm{R\ e}\left(\frac{1}{u}\right) > \frac{1}{\kappa}$, completing the proof.

Note: The function f on R is said to vanish below if there is a constant $c \in R$ such that $f(t) = 0$ for $t < c$. The set of functions that are locally integrable and vanish below will be denoted by loc_+. Most of the functions we shall be concerned in this paper vanish for $t < 0$.

Theorem 4: Let $\lambda > -1$ then

(1) If $f = loc_+$ and $\lim_{t\to\infty}\left[\frac{f(t)}{t^\lambda}\right]$ exists, so does $\lim_{\frac{1}{u}\to 0+}\left[\frac{S[f(t);u]}{u^{\lambda+1}}\right]$.

and we have $\lim_{t\to\infty}\left[\frac{f(t)}{t^\lambda}\right] = \frac{1}{\Gamma(\lambda+1)}\lim_{\frac{1}{u}\to 0+}\left[\frac{S[f(t);u]}{u^{\lambda+1}}\right]$.

(2) If f is Sumudu transformable and satisfies $f(t) = 0$ for $t < 0$ and if $\lim_{t\to 0+}\left[\frac{f(t)}{t^\lambda}\right]$ also $\lim_{\frac{1}{u}\to\infty}\left[\frac{S[f(t);u]}{u^{\lambda+1}}\right]$ and we have $\lim_{t\to 0+}\left[\frac{f(t)}{t^\lambda}\right] = \frac{1}{\Gamma(\lambda+1)}\lim_{\frac{1}{u}\to\infty}\left[\frac{S[f(t);u]}{u^{\lambda+1}}\right]$.

Proof:

(1) Let $\frac{f(t)}{t} \to \alpha$ as $t \to \infty$. This implies that there are constants A and $\rho > 0$ such that $\frac{|f(t)|}{t^{\lambda}} \le A$ for $t > \rho$. This further implies that $e^{\frac{-t}{u}} f(t)$ is integrable for all $\frac{1}{u} > 0$ so that we may write, if $f(t) = 0$ for $t < c$,

$$\begin{aligned} S[f(t);u] &= \int_c^{\infty} e^{-\frac{1}{u}} f(t)\,dt \\ &= \int_c^{\rho} e^{-\frac{1}{u}} f(t)\,dt + \int_{\rho}^{\infty} e^{-\frac{1}{u}} f(t)\,dt. \end{aligned} \qquad (1)$$

It is easy to see that $\frac{1}{u^{\lambda+1}}$ time the first term on the right of Equation (1) tends to zero as $\frac{1}{u} \to 0+$. Then $\frac{1}{u^{\lambda+1}}$ times the second term on the right of Equation (1) may be written as follows $\frac{1}{u^{\lambda}} \int_{\frac{\rho}{u}}^{\infty} e^{-x} f(ux)\,dx = \int_{\frac{\rho}{u}}^{\infty} x^{\lambda} e^{-x} \frac{f(ux)}{(ux)^{\lambda}}\,dx$ as $\frac{1}{u} \to 0+, \frac{f(ux)}{(ux)^{\lambda}}$ tends to α and since it is bounded in the range of integration by the constant A, we may apply the dominated convergence theorem and conclude that $\frac{1}{u^{\lambda+1}} S[f(t)](u) \to \int_{\frac{\rho}{u}}^{\infty} x^{\lambda} e^{-x} \alpha\,dx = \alpha\Gamma(\lambda+1)$ as $\frac{1}{u} \to 0+$ which complete the proof.

(2) Let $\frac{f(t)}{t^{\lambda}} \to \beta$ as $t \to 0+$. Since this function bounded in a neighbourhood of zero then there are constants B and $\sigma > 0$ such that $\frac{|f(t)|}{t^{\lambda}} \le B$ for $0 < t < \sigma$. Using a method similar to that in the proof of theorem (3) we let $f_0 = f(t)\left[1 - H(t-\sigma)\right]$ and $f_1(t) = f(t) H(t-\sigma)$.

Then

$$S\left[f(t);u\right]=\int_0^{\sigma} e^{\frac{-t}{u}} f(t)\,dt + S\left[f_1(t);u\right] \tag{2}$$

for $\frac{1}{u}\in domS\left[f\right]$ and apply $\left|S\left[f(t);u\right]\right|\le Ae^{-\frac{c}{u}}$ then we have $\left|S\left[f_1(t);u\right]\right|\le Ke^{-\frac{\sigma}{u}}$ for some constant K and $\frac{1}{u}$ sufficiently large, further $\frac{1}{u^{\lambda+1}}S\left[f_1(t);u\right]\to 0$ as $\frac{1}{u}\to\infty$. Also, by a similar argument to that was used in (1) $\frac{1}{u^{\lambda+1}}$ times the first term on the right hand side of equation (2) tends to $\beta\,\Gamma(\lambda+1)$ as $\frac{1}{u}\to\infty$, which completes the proof.

Now we let, f be a locally integrable function on R. We shall say that f is convergent (rather than integrable) if there is a constant k such that, for each, $\omega\in D$, $\lim\limits_{\lambda\to\infty}\left(\int\omega\left(\frac{t}{\lambda}\right)f(t)\,dt\right)$ exists and equal $k\omega(0)$ (where λ tends to infinity through) real values greater than zero. The constant k we shall denote by $\int f(t)\ dt$.

Other notations might also be used, for example if $f(t)=0$ for $t<0$, $\int f(t)\ dt$ will also be written as $\int_0^{\infty} f(t)\ dt$.

That means $\int f(t)\ dt=\lim\limits_{\lambda\to\infty}\left(\int\omega\left(\frac{t}{\lambda}\right)f(t)\,dt\right)$ for any $\omega\in D$ such that, $\omega(0)=1$, see Guest (1991).

Proposition 1: (Sumudu transform of derivative)

(1) Let f be differentiable on $(0,\infty)$ and let $f(t)=0$ for $t<0$. Suppose that $f'\in L_{loc}$. Then $f'\in L_{loc}$, $dom\ (Sf)\subset dom(f')$ and $S(f')=\frac{1}{u}S\left[f(t);\ u\right]-\frac{1}{u}f(0+)$.

(2) For $u \in dom\ S(f)$. More generally, if f is differentiable on (c,∞), the function $f(t)=0$ for $t<0$ and $f' \in L_{loc}$ then

$$S\left[f'(t);\ u\right]=\frac{1}{u}S\left[f(t);\ u\right]-\frac{1}{u}e^{-\frac{c}{u}}f(c+) \text{ for } u\in dom\ S(f).$$

Proof: We start by (2) as follows, the local integrability implies that $f(c+)$ exists since if $x>c$,

$$f(x)=f(c+1)-\int_x^{c+1}f'(t)\,dt \to f(c+1)-\int_x^{c+1}f'(t)\,dt \quad \text{as } x\to c+.$$

Let $u\in dom\left(S\left[f(t);\ u\right]\right)$. If $\omega\in D$ on using the integration by parts we have

$$\frac{1}{u}\int\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f'(t)dt=\frac{1}{u}\int_c^\infty\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f'(t)\,dt$$

$$=\lim_{x\to c+}\left[\frac{1}{u}\int_c^\infty\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f'(t)\,dt\right]$$

$$=\lim_{x\to c+}\left[\frac{1}{u}\omega\left(\frac{t}{\lambda}\right)e^{-\frac{t}{u}}f(t)\right]_x^\infty$$

$$-\lim_{x\to c+}\left(\frac{1}{u}\int_c^\infty e^{-\frac{t}{u}}\left[\frac{t}{\lambda}\omega'\left(\frac{t}{\lambda}\right)-\frac{1}{u}\omega\left(\frac{t}{\lambda}\right)\right]f(t)\,dt\right)$$

$$=\lim_{x\to c+}\left[-\frac{1}{u}\omega\left(\frac{t}{\lambda}\right)e^{-\frac{x}{u}}f(x)\right]$$

$$-\frac{1}{u}\left(\int_c^\infty e^{-\frac{t}{u}}\left[\frac{t}{\lambda}\omega'\left(\frac{t}{\lambda}\right)-\frac{1}{u}\omega\left(\frac{t}{\lambda}\right)\right]f(t)\,dt\right).$$

The first term on the right hand side is given by $-\frac{1}{u}\omega\left(\frac{c}{\lambda}\right)e^{-\frac{c}{u}}f(c+)$ which tends to $-\frac{1}{u}\omega(0)e^{-\frac{c}{u}}f(c+)$ as $\lambda\to\infty$. Then the second term is given by

$$-\frac{1}{u\lambda}\int e^{-\frac{t}{u}}\omega'\left(\frac{t}{\lambda}\right)f(t)\,dt+\frac{1}{u^2}\int e^{-\frac{t}{u}}\omega\left(\frac{t}{\lambda}\right)f(t)\,dt,$$

which tends to $0+\frac{1}{u}\omega(0)S(f)$ as $\lambda\to\infty$.

We have thus proved that for any $\omega\in D$,

$$\lim_{\lambda\to\infty}\left[\frac{1}{u}\int\omega\left(\frac{t}{\lambda}\right)\ e^{-\frac{t}{u}}f'(t)dt\right]$$
$$=\frac{\omega(0)}{u}\left[S(f)-f(c+)\right].$$

This implies that $e^{-\frac{t}{u}}f'$ is convergent, that is, $u\in dom\left(S\left[f(t);\ u\right]\right)$, and that

$$S\left[f'(t);\ u\right]=\frac{1}{u}S\left[f(t);\ u\right]-\frac{1}{u}e^{-\frac{c}{u}}f(c+)$$

In order to prove part (1), we just replace c by zero.

In general case, if f be differentiable on (a,b) with $a<b$ and $f(t)=0$ for $t<a$ or $t>b$ and $f'\in L_{loc}$ then, for all u

$$S\left[f'(t);\ u\right]=\frac{1}{u}S\left[f(t);\ u\right]$$
$$-\frac{1}{u}e^{-\frac{a}{u}}f(a+)+\frac{1}{u}e^{-\frac{b}{u}}f(b-).$$

In the next example we use differential equation and Sumudu transform.

Example: Let $y(t)=\sinh\left(\sqrt{t}\right)$. The Sumudu transform of $f=y$ is required. We first obtain a differential equation satisfied by y. We have

$$y'(t)=\frac{\cosh\left(\sqrt{t}\right)}{2\sqrt{t}}.$$

Hence

$$ty'(t) = \frac{\sqrt{t}}{2}\cosh\left(\sqrt{t}\right).$$

Therefore

$$\frac{d}{dt}\left[ty'(t)\right] = \frac{1}{4}\sinh\left(\sqrt{t}\right) + \frac{1}{4\sqrt{t}}\cosh\left(\sqrt{t}\right)$$
$$\frac{y(t)}{4} + \frac{y'(t)}{2}, \;\; t > 0.$$

If we now write $f = y$ we shall have, for any $t \neq 0$,

$$\frac{d}{dt}\left[tf'(t)\right] = \frac{f(t)}{4} + \frac{f'(t)}{2}. \tag{3}$$

Now by taking Sumudu transform of equation (3) we have

$$\frac{1}{u}S\left[tf'(t);\; u\right] - k = \frac{S\left[tf(t);\; u\right]}{4} + \frac{S\left[f'(t);\; u\right]}{2}$$

where $k = \lim_{t\to 0+}\left[tf'(t)\right]$. Clearly $f(0) = 0$ and $k = \lim_{t\to 0+}\left[\frac{\sqrt{t}}{2}\cosh(\sqrt{t})\right] = 0,$

follows that

$$\frac{1}{u}S\left[tf'(t);\; u\right] = \frac{S\left[f(t);\; u\right]}{4} + \frac{1}{2u}S\left[f(t);\; u\right]. \tag{4}$$

Now on using the proposition (1) the left hand side of equation (4) becomes

$$u\frac{d}{du}\left[S\left[f'(t);\; u\right]\right] = \frac{S\left[f(t);\; u\right]}{4} + \frac{1}{2u}S\left[f(t);\; u\right].$$

by simplification we have

$$\frac{F'(u)}{F(u)} = \frac{1}{4} + \frac{3}{2u}.$$

That is

$$F(u) = C'e^{\frac{1}{4}u}\sqrt{u^3}.$$

Now by replacing u by $\frac{1}{s}$ and in order to find the value of C' we apply the formula

$$\lim_{t\to 0+}\frac{f(t)}{\sqrt{t}} = \lim_{t\to 0+}\frac{\sinh\sqrt{t}}{\sqrt{t}} = 1$$

therefore

$$F\left(\frac{1}{s}\right) = C'e^{\frac{1}{4s}}\sqrt{\left(\frac{1}{s}\right)^3}$$

then

$$\lim_{s\to\infty}\frac{s^{\frac{3}{2}}F\left(\frac{1}{s}\right)}{\Gamma\left(\frac{3}{2}\right)} = 1$$

since

$$\frac{s^{\frac{3}{2}}F\left(\frac{1}{s}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{C'}{\Gamma\left(\frac{3}{2}\right)}e^{\frac{1}{4s}} \to \frac{C'}{\Gamma\left(\frac{3}{2}\right)}$$

as $s\to\infty$ and we have $C' = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$, thus finally we obtain

$$S\left[\sinh\left(\sqrt{t}\right);\ u\right] = \frac{\sqrt{\pi}}{2s^{3/2}}e^{\frac{1}{4s}}.$$

Before we extend proposition (1) to higher derivatives, we introduce the following notation: Let $P(x) = \sum_{k=0}^{n}\frac{a_k}{x^k}$ be a polynomial in x, where $n \geq 0$ and $a_n \neq 0$. We define $M_p(x)$ to be the $1\times n$ matrix of polynomials which is given by the following matrix product:

$$M_p(x) = \left(\frac{1}{x} \frac{1}{x^2} \frac{1}{x^3} \cdots \frac{1}{x^{n-1}} \right) \begin{pmatrix} a_1 & a_2 & . & . & . & a_n \\ a_2 & a_3 & . & . & a_n & 0 \\ a_3 & . & . & a_n & 0 & 0 \\ . & . & . & . & . & . \\ a_n & 0 & . & . & . & 0 \end{pmatrix} \quad (5)$$

For each complex number x, $M_p(x)$ define a linear mapping of C^n into C in obvious way. We shall write vectors y in C^n as row vectors or column vectors interchangeably, whichever is convenient, when $M_p(x)y$ is to be computed and the matrix representation by equation (5) of $M_p(x)$ is used, then of course y must be written as a column vector.

$M_p(x)y = \sum_{i=1}^{n} \frac{1}{x^i} \sum_{k=0}^{n-i} a_{i+k} y_k$ for any $y = (y_0, y_1, ..., y_{n-1}) \in \mathrm{C}^n$. If $n = 0$, $M_p(x)$ is defined to be unique linear mapping of $\{0\} = \mathrm{C}^0$ into C (empty matrix). In general, if $n > 0$ and f is $n-1$ times differentiable on an interval (a,b) with $a < b$, we shall write,

$$\varphi(f;a;n) = \left(f(a+), f'(a+), ..., f^{(n-1)}(a+) \right) \in \mathrm{C}^n$$

and

$$\phi(f;b;n) = \left(f(b-), f'(b-), ..., f^{(n-1)}(b-) \right) \in \mathrm{C}^n.$$

If $a = 0$, we write $\varphi(f;n)$ for $\varphi(f;0;n)$. If $n = 0$, we define $\varphi(f;a;0) = \phi(f;a;0) = 0 \in \mathrm{C}^0$.

In the next we recall the Sumudu transform of higher derivatives.

Proposition 2: (Sumudu transform of higher derivatives): Let f be n times differentiable on $(0,\infty)$ and let $f(t) = 0$ for $t < 0$. Suppose that $f^{(n)} \in L_{loc}$.

Then $f^{(k)} \in L_{loc}$ for $0 \le k \le n-1$, $domS\left[f(t);u\right] \subset dom\left(S\left[f^{(n)}(t);u\right]\right)$ and, for any polynomial P of degree n,

$$P(u)S(y)(u) = S(f)(u) + M_p(u)\varphi(y,n) \tag{6}$$

for $u \in dom\left(S\left[f(t);u\right]\right)$. In particular

$$S\left[f^{(n)}(t);u\right] = \frac{1}{u^n}S\left[f(t);u\right] - \left(\frac{1}{u^n},\frac{1}{u^{n-1}},\ldots,\frac{1}{u}\right)\varphi(f;n) \tag{7}$$

(with $\varphi(f;n)$ here written as a column vector). For $n=2$ we have

$$S\left[f''(t);u\right] = \frac{1}{u^2}S\left[f(t);u\right] - \frac{1}{u^2}f(0+) - \frac{1}{u}f'(0+). \tag{8}$$

Proof:

We use induction on *n*. The result is trivially true if $n=0$ and the case $n=1$ is equivalent to proposition (1). Suppose now that the result is true for some $n \geq 1$ and let

$$p(x) = \sum_{k=0}^{n+1}\frac{a_k}{x^k},$$

having a degree $n+1$. The first two statements follow by putting $z = f'$ and using the induction hypothesis and proposition (1). Now write $P(x) = a_0 + \frac{1}{x}W(x)$, where

$$W(x) = \sum_{k=0}^{n}\frac{a_{k+1}}{x^k}.$$

Then $P(D)f = a_0 f + W(D)z$ and therefore

$$\begin{aligned} S\left[P(D)f(t);u\right] &= a_0 S\left[f(t);u\right] + S\left[W(D)z\right](u) - M_w(u)\varphi(z;n) \\ &= a_0 S\left[f(t);u\right] + W(u)\left[\frac{1}{u}S\left[f(t);u\right] - \frac{1}{u}f(0+)\right] \\ &\quad - \sum_{i=1}^{n}\frac{1}{u^i}\sum_{k=0}^{n-i}a_{i+k+1}f^{(k+1)}(0+) \end{aligned}$$

on using equation (7) and $z^{(k)} = f^{(k+1)}$. The above summation can be written in the form of

$$\sum_{i=1}^{n}\frac{1}{u^{i}}\sum_{k=1}^{n-i+1}a_{i+k}f^{(k)}(0+)=\sum_{i=1}^{n}\frac{1}{u^{i}}\sum_{k=0}^{n-i+1}a_{i+k}f^{(k)}(0+)-\sum_{i=1}^{n}\frac{1}{u^{i}}a_{i}f(0+)$$

$$=\sum_{i=1}^{n+1}\frac{1}{u^{i}}\sum_{k=0}^{n-i+1}a_{i+k}f^{(k)}(0+)$$

$$-\frac{1}{u}\left[\frac{1}{u^{n}}a_{n+1}f(0+)+\sum_{i=1}^{n}\frac{1}{u^{i-1}}a_{i}f(0+)\right]$$

$$=M_{p}(u)\varphi(f;n)-\frac{1}{u}W(u)f(0+).$$

Thus

$$S\left[P(D)f\right](u)=\left[a_{0}+W(u)\frac{1}{u}\right]S\left[f(t);u\right]$$

$$-\frac{1}{u}W(u)f(0+)-M_{p}(u)\varphi(f;n)+\frac{1}{u}W(u)f(0+)$$

$$=P(u)S\left[f(t);u\right]-M_{p}(u)\varphi(f;n).$$

In general, if f is differentiable on (a,b) with $a<b$, and $f(t)=0$ for $t<a$ or $t>b$ and $f(n)\in L_{loc}$ then, for all u

$$S\left[P(D)f\right](u)=P(u)S\left[f(t);u\right]-M_{p}(u)\left[e^{-\frac{a}{u}}\varphi(f;b;n)\right].$$

In particular, if we consider $y(t)=\sin(t)$ clearly $y''+y=0$ and put $f=yH$, then also $f''+f=0$. Thus

$$\left(D^{2}+1\right)f=0.$$

Since $dom(Sf)$ contain $(0,\infty)$ we have from equation (6) and equation (7) with $n=2$ and $P(x)=x^{2}+1$, for $u>0$,

$$0=\left(\frac{1}{u^{2}}+1\right)S(f)-\left(\frac{1}{u}\,\frac{1}{u^{2}}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Since $\varphi(y,2)=(f(0),f'(0))=(0,1)$. Thus from above equation we get

$$S\left[\sin(t)H(t);u\right]=\frac{u}{u^2+1}.$$

Let y be n times differentiable on $(0,\infty)$, zero on $(-\infty,0)$ and satisfy the following equation

$$P(D)y=f*g \tag{9}$$

under the initial condition

$$y(0)=y_0, y'(0)=y_1,..., y^{(n-1)}(0)=y_{n-1}.$$

Then $y^{(k)}$ is locally integrable and Sumudu transformable for $0\le k\le n$ and for every such k, then Sumudu transform of equation (9) given by equation (6) where

$$P(u)=\frac{a_n}{u^n}+\frac{a_{n-1}}{u^{n-1}}+...+a_0,$$

$$M_p(u)\varphi(y,n)=\left(\frac{1}{u}\frac{1}{u^2}\cdots\frac{1}{u^n}\right)\begin{pmatrix} a_1 & a_2 & . & . & . & a_n \\ a_2 & a_3 & . & . & a_n & 0 \\ a_3 & . & . & a_n & 0 & 0 \\ . & . & . & . & . & . \\ a_n & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{n-1} \end{pmatrix}$$

where * indicates the single convolution. In particular for, $n=2$ we have

$$\left(\frac{a_2}{u^2}+\frac{a_1}{u}a_0\right)S\left[y(t);u\right]=S(f*g)(u)+\left(\frac{1}{u}\frac{1}{u^2}\right)\begin{pmatrix} a_1 & a_2 \\ a_2 & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

In order to get the solution of Equation (9), we taking inverse Sumudu transform for Equation (6) as follows

$$y(t)=S^{-1}\left[\frac{(f*g)(u)}{P(u)}\right]+S^{-1}\left[\frac{M_P(u)}{P(u)}\varphi(y,n)\right] \tag{10}$$

we assume that the inverse exist for each terms in the right hand side of Equation (9).

Now, let us multiply the right hand side of Equation (9) by polynomial $\Psi(t)=\sum_{k=0}^{n} t^k$, we obtain the non constant coefficients in the form of

$$\Psi(t)*\left[P(D)y\right]=f*g \tag{11}$$

under the same initial conditions used above. By taking Sumudu transform for equation (11) and using the initial condition, after arrangement we have

$$S\left[y(t);u\right]=\frac{F(u)G(u)}{k!u^k P(u)}$$

$$+\frac{1}{P(u)}\left(\frac{1}{u}\frac{1}{u^2}\cdots\frac{1}{u^{n+1}}\right)\begin{pmatrix} a_1 & a_2 & . & . & . & a_n \\ a_2 & a_3 & . & . & a_n & 0 \\ a_3 & . & . & a_n & 0 & 0 \\ . & . & . & . & . & . \\ a_n & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{n-1} \end{pmatrix}$$

by taking inverse Sumudu transform we have

$$y=S^{-1}\left[\frac{F(u)G(u)}{k!u^k P(u)}+\frac{1}{P(u)}\left(\frac{1}{u}\frac{1}{u^2}\cdots\frac{1}{u^{n+1}}\right)\right]\begin{pmatrix} a_1 & a_2 & . & . & . & a_n \\ a_2 & a_3 & . & . & a_n & 0 \\ a_3 & . & . & a_n & 0 & 0 \\ . & . & . & . & . & . \\ a_n & 0 & . & . & . & 0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{n-1} \end{pmatrix} \tag{12}$$

here we assume the inverse exist.

Now, if we substitute Equation (12) into Equation (11) we obtain the non homogeneous term of Euation (11) $f*g$ and polynomial in the form of $\Phi(t)=-\sum_{k=1}^{n}\frac{1}{k!}t^k$.

Thus we note that the convolution product can be used to generate the differential equations with variable coefficients.

2. CONCLUSION

In this study the applications of the Sumudu transform to the solution of differential equations with constant and non-constant coefficients have been demonstrated.

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